

# A simple rank-based Markov chain with self-organized criticality

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## Abstract

We introduce a self-reinforced point processes on the unit interval that appears to exhibit self-organized criticality, somewhat reminiscent of the well-known Bak Sneppen model. The process takes values in the finite subsets of the unit interval and evolves according to the following rules. In each time step, a particle is added at a uniformly chosen position, independent of the particles that are already present. If there are any particles to the left of the newly arrived particle, then the left-most of these is removed. We show that all particles arriving to the left of  $p_c = 1 - e^{-1}$  are a.s. eventually removed, while for large enough time, particles arriving to the right of  $p_c$  stay in the system forever.

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# 1 Introduction and results

## 1.1 Main results

Let  $(U_k)_{k \geq 1}$  be an i.i.d. collection of uniformly distributed  $[0, 1]$ -valued random variables. For each finite subset  $x$  of  $[0, 1]$ , we inductively define a sequence  $X^x = (X_k^x)_{k \geq 0}$  of random finite subsets of  $[0, 1]$  by  $X_0^x := x$ ,  $M_{k-1}^x := \min(X_{k-1}^x \cup \{1\})$  and

$$X_k^x := \begin{cases} X_{k-1}^x \cup \{U_k\} & \text{if } U_k < M_{k-1}^x, \\ (X_{k-1}^x \cup \{U_k\}) \setminus \{M_{k-1}^x\} & \text{if } U_k > M_{k-1}^x. \end{cases} \quad (k \geq 1). \quad (1)$$

In words, this says that  $M_{k-1}^x$  is the minimal element of  $X_{k-1}^x$  and that the set  $X_k^x$  is constructed from  $X_{k-1}^x$  by adding  $U_k$ , and in case that  $M_{k-1}^x < U_k$ , removing  $M_{k-1}^x$  from  $X_{k-1}^x$ . Since the  $(U_k)_{k \geq 1}$  are i.i.d. and  $X_k^x$  is a function of  $X_{k-1}^x$  and  $U_k$ , it is clear that  $X^x$  is a Markov chain. (In fact, we have just given a *random mapping representation* for it.) The state space of  $X^x$  is the set  $\mathcal{P}_{\text{fin}}[0, 1]$  of all finite subsets of  $[0, 1]$ , which is naturally isomorphic to the space of all simple counting measures on  $[0, 1]$  (i.e.,  $\mathbb{N}$ -valued measures  $\nu$  such that  $\nu(\{x\}) \leq 1$  for all  $x \in [0, 1]$ ). We equip this space with the topology of weak convergence and the associated Borel- $\sigma$ -algebra.

The process  $X^x$  is an example of a Markov process with self-reinforcement (compare [Pem07]), since the number of particles in the system can grow without bounds and influences the fate of newly arrived particles. As we will see in a moment, it also appears to exhibit self-organized criticality in a way that is reminiscent of the well-known Bak Sneppen model. The empirical distribution function  $F_k^x(q) := |X_k^x \cap [0, q]|$  can loosely be interpreted as the profile of a canyon being cut out by a river. If  $U_k < M_{k-1}^x$ , then the river cuts deeper into the rock. If  $U_k > M_{k-1}^x$ , then the slope of the canyon between  $U_k$  and the river is eroded one step down.

Our first result says that particles arriving on the left of the critical point  $p_c := 1 - e^{-1}$  are eventually removed from the system, but for large enough time, particles arriving on the right of  $p_c$  stay in the system forever.

**Theorem 1 (A.s. behavior of the minimum)** *For any finite  $x \subset [0, 1]$ , one has*

$$\limsup_{k \rightarrow \infty} M_k^x = 1 - e^{-1} \quad \text{a.s.} \quad (2)$$

To understand Theorem 1 better, note that for each  $0 \leq q \leq 1$ , the restriction  $X_k^x \cap [0, q]$  of  $X_k^x$  to  $[0, q]$  is a Markov chain. Indeed, particles arriving on the right of  $q$  just have the effect that in each time step, with probability  $1 - q$ , the minimal element of  $X_k^x \cap [0, q]$ , if there is one, is removed, while no new particles are added inside  $[0, q]$ . Theorem 1 says that this Markov chain is recurrent for  $q < p_c$  and transient for  $q > p_c$ . For any  $q \in [0, 1]$ , let

$$\tau_q^\emptyset := \inf\{k > 0 : X_k^\emptyset \cap [0, q] = \emptyset\} \quad (3)$$

be the first time the restricted process  $X_k^\emptyset \cap [0, q]$  returns to the empty set. Our next theorem shows that for  $q < p_c$ , the restricted chain is positively recurrent and ergodic, while for  $q > p_c$ , it is transient. Below, we call a subset of  $[0, p_c)$  *locally finite* if its intersection with any compact subset of  $[0, p_c)$  is finite.

**Theorem 2 (Ergodicity of restricted process)** *Let  $p_c := 1 - e^{-1}$ . Then*

$$\mathbb{E}[\tau_q^\emptyset] = (1 + \log(1 - q))^{-1} \quad (q < p_c) \quad \text{and} \quad \mathbb{P}[\tau_q^\emptyset = \infty] > 0 \quad (q > p_c). \quad (4)$$

*Moreover, there exists a random, locally finite subset  $X_\infty \subset [0, p_c)$  such that, regardless of the initial state  $x$ ,*

$$\mathbb{P}[X_k^x \cap [0, q] \in \cdot] \xrightarrow[k \rightarrow \infty]{} \mathbb{P}[X_\infty \cap [0, q] \in \cdot] \quad (0 < q < p_c), \quad (5)$$

*where  $\rightarrow$  denotes convergence of probability measures in total variation norm distance. The random point set  $X_\infty$  a.s. consists of infinitely many points.*

Numerical simulations strongly suggest that at  $q = p_c$ , the restricted chain  $X_k^x \cap [0, q]$  is null recurrent and, starting from a state with no particles on the left on  $q$ , the probability that one has to wait longer than  $k$  steps before the area on the left of  $q$  is again empty decays as  $k^{-1/2}$ , but we have no proof for this. Note that such a proof would establish self-organized criticality for our process. Our process is self-organized in the sense that it finds the transition point  $p_c$  by itself. In particular, one does not have to tune a parameter of the model to exactly the right value to see the (presumed) power-law critical behavior at  $p_c$ .

## 1.2 Discussion

Our model is similar to the well-known Bak Sneppen model [BS93], which is one of the best-known models exhibiting self-organized criticality, although this is only been fully rigorously established for a simplified version of the model [MS12]. Like our process, the Bak Sneppen model and its modifications are also based on the principle that the particle with the lowest value is killed. This rule alone, however, is not enough to see interesting behavior.

In our process, we add particles one by one and also kill the particle with the lowest value, but only if this is not the newly arrived particle. In this way, the total population is allowed to grow and the process takes the limit of large population size by itself, so to say. In the Bak Sneppen model, the total number of particles is fixed, and when a particle is killed, not only this particle, but also some of its neighbors (according to some additional structure) are killed, and the killed particles are replaced by new particles with uniformly chosen values. The original Bak Sneppen model and its modifications differ in the way these “neighbors” are chosen. In the original model, the particles are numbered  $0, \dots, N - 1$  and their neighbors are those with neighboring numbers (modulo  $N$ ). In the modified model from [MS12], one “neighbor” is chosen uniformly from the population, with a new choice for each time step.

Closely related to the Bak Sneppen model is the Barabási queueing system introduced in [Bar05], which has so far been studied only in the physics literature. Exact results for this model have been derived in [Vaz05, Ant09]. In the original model, items in a queue have a priority taking values in a continuous interval. In each time step, with probability  $p$  close to one, the item with the highest priority is served, and with the remaining probability a random item is removed from the list. At the end of each step, a new item is added so that the length of the queue remains constant.

In [CG09], this latter assumption is dropped and the number of items added in each time step is assumed to be larger than one, with the result that some items never get served and the length of the queue grows without bounds, in a way that is very similar to our model. They show that their model can be mapped to invasion percolation on a tree. Using this mapping,

they are able to prove power-law behavior at the critical point. A similar mapping also exists for Barabási's original model [CG07]. Contrary to our model, the critical point for the model in [CG09] is trivial since the number of items added and removed in each step is known.

Somewhat similar in spirit to these models is also the model [GMS11], which is basically a supercritical branching process in which fitnesses are assigned to the particles and those killed have the lowest fitness.

We note that in the construction of all these processes and in particular also ours, only the relative order of the points (i.e., their rank or priority) matters, so replacing the uniform distribution on  $[0, 1]$  by any other atomless law on  $\mathbb{R}$  yields the same model up to a continuous transformation of space. Starting from the empty initial state, adding points one by one, and taking notice only of their relative order, one in effect constructs after  $k$  steps a random permutation of  $k$  elements. In view of this, our quantities of interest may be described as functions of such a random permutation. This is somewhat reminiscent of the way the authors of [AD99] use what they call Hammersley's process to study the longest increasing subsequence of a random permutation. There is an extensive literature on functions of random permutations, but none of those studied so far seem relevant for our process.

## 2 Proofs

### 2.1 Main idea of the proofs

In the present section, we describe the main idea of the proof of Theorems 1 and 2. As already mentioned in Section 1.2, by a simple transformation of space, we may replace the uniformly distributed random variables  $(U_k)_{k \geq 1}$  by real random variables having any non-atomic distribution. At present, it will be more convenient to work with exponentially distributed random variables with mean one, so we transform the unit interval  $[0, 1]$  into the closed halfline  $[0, \infty]$  with the transformation  $q \mapsto f(q) := -\log(1 - q)$ , set  $\sigma_k := f(U_k)$  ( $k \geq 1$ ), and, concentrating for the moment on the process started in the empty initial state, we let  $Y_k := f(X_k^\emptyset)$  denote the image of  $X_k^\emptyset$  under  $f$ . Then

$$Y_k := \begin{cases} Y_{k-1} \cup \{\sigma_k\} & \text{if } \sigma_k < N_{k-1}, \\ (Y_{k-1} \cup \{\sigma_k\}) \setminus \{N_{k-1}\} & \text{if } \sigma_k > N_{k-1}. \end{cases} \quad (k \geq 1), \quad (6)$$

where  $N_k := \min(Y_k \cup \{\infty\})$ . Let

$$F_t(k) := |Y_k^{(t)}| \quad \text{with} \quad Y_k^{(t)} := Y_k \cap [0, t] \quad (k \geq 0, t \geq 0) \quad (7)$$

denote the number of points on the left of  $t$ .

We claim that the function-valued process  $(F_t)_{t \geq 0}$  with  $F_t = (F_t(k))_{k \geq 0}$  is a continuous-time Markov processes, where the parameter  $t$  plays the role of time. Indeed, at each time  $t = \sigma_k$ , let  $F_{t-} := |Y_k \cap [0, t)|$  denote the state immediately prior to  $t$  and let

$$\kappa_t(k) := \inf\{k' > k : F_{t-}(k' - 1) = 0 \text{ and } \sigma_{k'} \geq t\}, \quad (8)$$

with the convention that  $\inf \emptyset := \infty$ . Then at the time  $t = \sigma_k$ , the function  $F_t$  changes as

$$F_t(k') = \begin{cases} F_{t-}(k') + 1 & \text{if } k \leq k' < \kappa_t(k), \\ F_{t-}(k') & \text{otherwise} \end{cases} \quad (k' \geq 0). \quad (9)$$

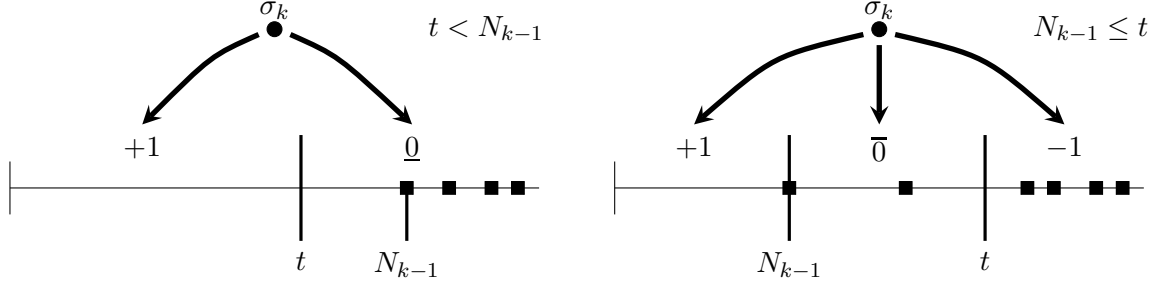


Figure 1: Illustration of the quantity  $\Delta F_t(k)$  from (11). The value of  $\Delta F_t(k)$  depends on the relative order of  $t$ ,  $N_{k-1}$ , and  $\sigma_k$ .

In the language of self-organized criticality, we may call such a move an avelange. In analogy with (3), let

$$\tilde{\tau}_t := \inf\{k > 0 : Y_k^{(t)} = \emptyset\} \quad (10)$$

denote the first time the restricted process  $Y^{(t)}$  returns to the empty set. At each (deterministic)  $t \geq 0$ , the function  $F_t$  starts in  $F_t(0) = 0$  and makes i.i.d. excursions away from 0 whose length is distributed as  $\tilde{\tau}_t$ .

We will be interested in the quantity

$$\Delta F_t(k) := \begin{cases} \underline{0} & \text{if } F_t(k) = F_t(k-1) = 0, \\ \bar{0} & \text{if } F_t(k) = F_t(k-1) > 0, \\ -1 & \text{if } F_t(k) = F_t(k-1) - 1, \\ +1 & \text{if } F_t(k) = F_t(k-1) + 1. \end{cases} \quad (k \geq 1, t \geq 0). \quad (11)$$

It follows from (9) that  $\Delta F_t$ , too, evolves in a Markovian way as a function of  $t$ . For each  $k \geq 1$ , at time  $t = \sigma_k$ , one has  $\Delta F_{t-}(k) \in \{\underline{0}, -1\}$  immediately prior to  $t$ , and the function  $\Delta F_t$  changes at time  $t$  according to the following rules.

- (i) If  $\Delta F_t(k) = \underline{0}$  prior to  $\sigma_k$ , then  $\Delta F_t(k)$  becomes  $+1$  at time  $\sigma_k$ .
- (ii) If  $\Delta F_t(k) = -1$  prior to  $\sigma_k$ , then  $\Delta F_t(k)$  becomes  $\bar{0}$  at time  $\sigma_k$ .
- (iii) In both previous cases, the next  $\underline{0}$  to the right of  $k$ , if there is one, becomes a  $-1$ .

These rules are further illustrated in Figure 1. Note that in these pictures, moving the level  $t$  up across the value of  $\sigma_k$ , the value of  $\Delta F_t(k)$  changes either from  $\underline{0}$  to  $+1$  or from  $-1$  to  $\bar{0}$ .

We observe that if  $(Y_k)_{k \geq 0}$  is defined in terms of  $(\sigma_k)_{k \geq 1}$  as in (6) and  $(Y_k^{(t)})_{k \geq 0}$  is the restricted process as in (7), then the joint process  $(Y_k^{(t)}, \sigma_k)_{k \geq 1}$  is a Markov chain. If for some  $t_+ > 0$ , the return time  $\tilde{\tau}_{t_+}$  from (10) has finite expectation, then it is not hard to see that this Markov chain (with  $t = t_+$ ) is ergodic, so it is possible to construct a stationary process  $(Y_k^{(t_+)}, \sigma_k)_{k \in \mathbb{Z}}$ , and such a process is unique in law. Setting

$$Y_k^{(t)} := Y_k^{(t_+)} \cap [0, t] \quad (0 \leq t \leq t_+) \quad (12)$$

we also obtain stationary Markov chains  $(Y_k^{(t)}, \sigma_k)_{k \in \mathbb{Z}}$  for all  $0 \leq t \leq t_+$ , and associated functions  $(F_t(k))_{k \in \mathbb{Z}}$ . We claim that for the stationary process, the densities of  $\underline{0}$ 's and  $-1$ 's

satisfy the following differential equations as a function of  $t$ , for  $0 \leq t \leq t_+$ :

$$\begin{aligned}\frac{\partial}{\partial t}\mathbb{P}[\Delta F_t(k) = \underline{0}] &= -2\mathbb{P}[\Delta F_t(k) = \underline{0}] - \mathbb{P}[\Delta F_t(k) = -1], \\ \frac{\partial}{\partial t}\mathbb{P}[\Delta F_t(k) = -1] &= \mathbb{P}[\Delta F_t(k) = \underline{0}].\end{aligned}\tag{13}$$

To see this, note that at a per site rate that is proportional to the density of  $\underline{0}$ 's, rules (i) and (iii) come into effect, leading to the disappearance of two  $\underline{0}$ 's and the creation of one  $-1$ . Similarly, at a per site rate that is proportional to the density of  $-1$ 's, rules (ii) and (iii) come into effect, leading to the disappearance of one  $\underline{0}$  and no net change in the number of  $-1$ 's. We can solve (13) with the initial condition  $\mathbb{P}[\Delta F_t(1) = \underline{0}] = 1$ ,  $\mathbb{P}[\Delta F_t(1) = -1] = 0$  explicitly to find

$$\mathbb{P}[\Delta F_t(1) = \underline{0}] = (1 - t)e^{-t} \quad \text{and} \quad \mathbb{P}[\Delta F_t(1) = -1] = te^{-t} \quad (0 \leq t \leq t_+).\tag{14}$$

Since the density of  $\underline{0}$ 's must be a nonnegative number, we see that no stationary process  $(Y_k^{(t_+)}, \sigma_k)_{k \in \mathbb{Z}}$  can exist for  $t_+ > 1$ . We will prove that on the other hand, for each  $t_+ \leq 1$ , a stationary process exists, and  $\tilde{\tau}_t$  has finite expectation for  $t < 1$ . Since the function  $F_t$  makes i.i.d. excursions away from 0 whose length is distributed as  $\tilde{\tau}_t$ , we can solve the expectation of  $\tilde{\tau}_t$  from the density of  $\underline{0}$ 's. Indeed, by a simple renewal argument, for each  $t < 1$ ,

$$\mathbb{P}[\Delta F_t(k) = \underline{0}] = \mathbb{E}[\tilde{\tau}_t]^{-1} \mathbb{P}[\tilde{\tau} = 1] = e^{-t} \mathbb{E}[\tilde{\tau}_t]^{-1}.\tag{15}$$

Combining this with (14), we find that

$$\mathbb{E}[\tilde{\tau}_t] = (1 - t)^{-1} \quad (t < 1).\tag{16}$$

Taking into account the transformation of variables  $t = f(q) := -\log(1 - q)$ , this yields the formula for  $\mathbb{E}[\tau_q^{\theta}]$  in (4).

## 2.2 An alternative approach

After the proofs of the present paper were written, I was made aware of a paper by Luckock [Luc03], who treats a generalization of a model due to Stigler [Sti64] modeling the evolution of bid and ask limit orders in an order book such as used on a stock market. Although this model differs considerably from our model, Luckock's methods can be adapted to the present setting, leading to a simpler differential equation than (13), from which the critical point can also be determined.

Since the approach sketched in the previous section also seems to have its merits, in particular, because of the observation that the process  $(\Delta F_t)_{t \geq 0}$  is a continuous-time Markov chain, we will stick to this approach in the proofs. For the interest of the reader, however, we sketch here the different approach based on Luckock's methods which seems simpler in certain aspects.

Assume, just as in the previous section, that for some  $t_+ > 0$ , the restricted process in  $Y^{(t_+)}$  is positively recurrent, let  $(Y_k^{(t_+)}, \sigma_k)_{k \in \mathbb{Z}}$  be a stationary process, and write  $N_k^{(t_+)} := \min(Y_k^{(t_+)} \cup \{t_+\})$  ( $k \in \mathbb{Z}$ ). For  $0 \leq t < t_+$  and  $k \in \mathbb{Z}$ , define as before

$$F_t(k) := |Y_k^{(t_+)} \cap [0, t]| \quad \text{and} \quad \Delta F_t(k) := F_t(k) - F_t(k - 1),\tag{17}$$

where in the present approach, it is not necessary to distinguish the cases  $\Delta F_t = \underline{0}$  and  $\Delta F_t = \bar{0}$ . Stationarity leads to the requirement that, for  $0 \leq t < t_+$ ,

$$\begin{aligned} \mathbb{P}[N_{k-1}^{(t_+)} \leq t] \mathbb{P}[\sigma_k > t] &= \mathbb{P}[\Delta F_t(k) = -1] \stackrel{!}{=} \mathbb{P}[\Delta F_t(k) = 1] \\ &= \mathbb{P}[\sigma_k < N_{k-1}^{(t_+)}, \sigma_k \leq t] = \int_0^t \mathbb{P}[\sigma_k \in ds] \mathbb{P}[s < N_{k-1}^{(t_+)}]. \end{aligned} \quad (18)$$

Using the fact that the  $\sigma_k$ 's are exponentially distributed with mean one, this yields

$$e^{-t}(1 - \mathbb{P}[t < N_{k-1}^{(t_+)}]) = \int_0^t e^{-s} ds \mathbb{P}[s < N_{k-1}^{(t_+)}] \quad (0 \leq t < t_+). \quad (19)$$

Since the right-hand side is differentiable with respect to  $t$ , the same must be true for the left-hand side, and one finds that

$$-e^{-t}(1 - \mathbb{P}[t < N_{k-1}^{(t_+)}]) - e^{-t} \frac{\partial}{\partial t} \mathbb{P}[t < N_{k-1}^{(t_+)}] = e^{-t} \mathbb{P}[t < N_{k-1}^{(t_+)}], \quad (20)$$

which can be simplified to

$$\frac{\partial}{\partial t} \mathbb{P}[t < N_{k-1}^{(t_+)}] = -1 \quad (0 \leq t < t_+). \quad (21)$$

Using the boundary condition  $\mathbb{P}[0 < N_{k-1}^{(t_+)}] = 1$ , one finds that  $\mathbb{P}[t < N_{k-1}^{(t_+)}] = 1 - t$  or equivalently, the expected time before the process restricted to  $[0, t]$  returns to the empty configuration is as in (16).

### 2.3 The lower invariant process

To make the arguments in Section 2.1 precise, we must show that the process  $(\Delta F_t(k))_{k \in \mathbb{Z}}$  in (13) is well-defined and that its one-dimensional distributions satisfy the differential equation (13) up to the first time that  $\mathbb{P}[\Delta F_t(k) = \underline{0}]$  hits zero. It is tempting to view  $(\Delta F_t)_{t \geq 0}$  as an interacting particle system, but since its jump rates do not satisfy the summability conditions of [Lig85, Thm. I.3.9], standard theory cannot be applied and we have to proceed differently.

Our first lemma says that solutions to the inductive formula (6) are monotone in the starting configuration.

**Lemma 3 (First comparison lemma)** *Let  $y$  and  $\tilde{y}$  be finite subsets of  $[0, 1]$  and let  $(Y_k)_{k \geq 0}$  and  $(\tilde{Y}_k)_{k \geq 0}$  be defined by the inductive relation (6) with  $Y_0 = y$  and  $\tilde{Y}_0 = \tilde{y}$ . Then  $y \subset \tilde{y}$  implies that  $Y_k \subset \tilde{Y}_k$  for all  $k \geq 0$ .*

**Proof** It suffices to show that  $Y_{k-1} \subset \tilde{Y}_{k-1}$  implies  $Y_k \subset \tilde{Y}_k$ . Adding the point  $\sigma_k$  to both  $Y_{k-1}$  and  $\tilde{Y}_{k-1}$  obviously preserves the order of inclusion, as does simultaneously removing the minimal elements  $N_{k-1}$  from  $Y_{k-1}$  and  $\tilde{N}_{k-1}$  from  $\tilde{Y}_{k-1}$ . Since  $Y_k \subset \tilde{Y}_k$  we have  $N_{k-1} \geq \tilde{N}_{k-1}$  and it may happen that  $\tilde{N}_{k-1} < \sigma_k \leq N_{k-1}$ , in which case we remove  $\tilde{N}_{k-1}$  from  $\tilde{Y}_{k-1}$  but not  $N_{k-1}$  from  $Y_{k-1}$ , but in this case  $\tilde{N}_{k-1}$  is not an element of  $Y_{k-1}$  so again the order is preserved. ■

We will be interested in stationary solutions to the inductive relation (6). To this aim, we consider a two-way infinite sequence  $(\sigma_k)_{k \in \mathbb{Z}}$  of i.i.d. exponentially distributed random variables with mean one. For each  $m \in \mathbb{Z}$ , we let  $(Y_{m,k})_{k \geq m}$  denote the solution to the inductive relation (6) started in  $Y_{m,m} := \emptyset$ . Since  $Y_{m-1,m} \supset \emptyset = Y_{m,m}$ , we see by Lemma 3,

that  $Y_{m-1,k} \supset Y_{m,k}$  for all  $k \geq m$ , so there exists a collection  $(Y_k)_{k \in \mathbb{Z}}$  of countable subsets of  $[0, \infty)$  such that

$$Y_{m,k} \uparrow Y_k \quad \text{as } m \downarrow -\infty. \quad (22)$$

We call the limit process  $(Y_k)_{k \in \mathbb{Z}}$  from (22) the *lower invariant process*. We set

$$F_t(k) := |Y_k \cap [0, t]| \quad (t \in [0, \infty), k \in \mathbb{Z}). \quad (23)$$

The following theorem is the main result of the present subsection.

**Theorem 4 (Lower invariant process)** *For all  $k \in \mathbb{Z}$ , one has*

$$F_t(k) \begin{cases} < \infty & \text{a.s.} & \text{if } t \in [0, 1), \\ = \infty & \text{a.s.} & \text{if } t \in [1, \infty). \end{cases} \quad (24)$$

For each  $k$ , the set  $Y_k$  a.s. has a minimal element  $N_k := \min(Y_k)$  and  $(Y_k)_{k \in \mathbb{Z}}$  solves the inductive relation (6) for all  $k \in \mathbb{Z}$ . Finally, one has

$$\begin{aligned} \mathbb{P}[\Delta F_t(k) = \underline{0}] &= (1-t)e^{-t}, \\ \mathbb{P}[\Delta F_t(k) = \overline{0}] &= 1 - (1+t)e^{-t}, \\ \mathbb{P}[\Delta F_t(k) = -1] &= te^{-t}, \\ \mathbb{P}[\Delta F_t(k) = +1] &= te^{-t}, \end{aligned} \quad (t \in [0, 1), k \in \mathbb{Z}), \quad (25)$$

where  $\Delta F_t(k)$  is defined as in (11).

As a first step towards the proof of Theorem 4, for  $t \in [0, \infty)$ , we look at the restricted lower invariant process

$$Y_k^{(t)} := Y_k \cap [0, t] \quad (k \in \mathbb{Z}). \quad (26)$$

We define  $N_k^{(t)} := \inf(Y_k^{(t)} \cup \{t\})$  ( $k \in \mathbb{Z}$ ).

**Lemma 5 (Restricted process)** *For each  $t \in [0, \infty)$ , one of the following two alternatives occurs:*

- (i)  $|Y_k^{(t)}| = \infty$  a.s. for all  $k \in \mathbb{Z}$ .
- (ii)  $|Y_k^{(t)}| < \infty$  a.s. for all  $k \in \mathbb{Z}$  and  $(Y_k^{(t)})_{k \in \mathbb{Z}}$  solves the inductive relation

$$Y_k^{(t)} := \begin{cases} Y_{k-1}^{(t)} \cup \{\sigma_k\} & \text{if } \sigma_k \leq N_{k-1}^{(t)}, \\ (Y_{k-1}^{(t)} \cup \{\sigma_k\}) \setminus \{N_{k-1}^{(t)}\} & \text{if } N_{k-1}^{(t)} < \sigma_k \leq t, \\ Y_{k-1}^{(t)} \setminus \{N_{k-1}^{(t)}\} & \text{if } t < \sigma_k, \end{cases} \quad (k \in \mathbb{Z}). \quad (27)$$

**Proof** Set  $Y_{m,k}^{(t)} := Y_{m,k} \cap [0, t]$ . Since  $|Y_{m,k-1}^{(t)}|$  and  $|Y_{m,k}^{(t)}|$  differ at most by one, letting  $m \rightarrow -\infty$ , we see that the event

$$\{|Y_{m,k}^{(t)}| = \infty \forall k \in \mathbb{Z}\} \cup \{|Y_{m,k}^{(t)}| < \infty \forall k \in \mathbb{Z}\} \quad (28)$$

has probability one. Since the indicators of both events occurring in this expression are translation invariant functions of the ergodic random variables  $(\sigma_k)_{k \in \mathbb{Z}}$ , it follows that exactly one of these events has probability one.



For each  $m \leq k-1$ , the sets  $Y_{m,k-1}$  and  $Y_{m,k}$  are related as in (6), which is easily seen to imply that  $Y_{m,k-1}^{(t)}$  and  $Y_{m,k}^{(t)}$  are related as in (27). If  $|Y_{k-1}^{(t)}| < \infty$ , then, since  $Y_{k-1}^{(t)}$  is the increasing limit of  $Y_{m,k-1}^{(t)}$  as  $m \rightarrow -\infty$ , there exists some  $m_0$  such that  $Y_{k-1}^{(t)} = Y_{m,k-1}^{(t)}$  for all  $m \leq m_0$ . Now also  $Y_k^{(t)} = Y_{m,k}^{(t)}$  for all  $m \leq m_0$  and hence  $Y_k^{(t)}$  and  $Y_{k-1}^{(t)}$  are related as in (27).  $\blacksquare$

**Lemma 6 (Minimality)** *Let  $t \in [0, \infty)$  be such that case (ii) of Lemma 5 holds, let  $(Y_k^{(t)})_{k \in \mathbb{Z}}$  be the restricted lower invariant process defined in (26) and let  $(\tilde{Y}_k^{(t)})_{k \in \mathbb{Z}}$  be any other solution of the two-way infinite inductive relation (27), taking values in the finite subsets of  $[0, t]$ . Then  $Y_k^{(t)} \subset \tilde{Y}_k^{(t)}$  for all  $k \in \mathbb{Z}$  a.s.*

**Proof** Set  $Y_{m,k}^{(t)} := Y_{m,k} \cap [0, t]$  as in the previous proof. Then  $(Y_{m,k}^{(t)})_{k \geq m}$  solves the inductive relation (27) for  $k \geq m$  and  $Y_{m,m}^{(t)} = \emptyset \subset \tilde{Y}_m^{(t)}$ . In analogy with Lemma 3, it is easy to prove that solutions to (27) are monotone in the initial state, so it follows that  $Y_{m,k}^{(t)} \subset \tilde{Y}_k^{(t)}$  for all  $k \geq m$ . Letting  $m \downarrow -\infty$  for fixed  $k$  now proves the statement.  $\blacksquare$

Since a.s.  $\sigma_k \neq 0$  for all  $k \in \mathbb{Z}$ , it is clear from the definition that  $Y_k \cap \{0\} = \emptyset$  and hence  $\Delta F_0(k) = \underline{0}$  for all  $k \in \mathbb{Z}$ . The next key proposition shows that  $F_t(k)$  is also a.s. finite for  $t$  small enough.

**Proposition 7 (Finite regime)** *Let  $s \in [0, \infty)$  and assume that a.s.  $F_s(k) < \infty$  for all  $k \in \mathbb{Z}$ . Then there exists some  $\varepsilon$  such that*

$$\mathbb{P}[\Delta F_s(k) = \underline{0}] = \varepsilon > 0 \quad (k \in \mathbb{Z}). \quad (29)$$

Moreover, for all  $t \geq s$  such that  $2(e^{-s} - e^{-t}) < \varepsilon$ , one has  $F_t(k) < \infty$  for all  $k \in \mathbb{Z}$  a.s. and

$$\mathbb{P}[\Delta F_t(k) = \underline{0}] \geq \varepsilon - 2(e^{-s} - e^{-t}) \quad (k \in \mathbb{Z}). \quad (30)$$

**Proof** By assumption,  $F_s(k)$  is a.s. finite, so  $\mathbb{P}[F_s(k) \leq m] > 0$  for some  $m < \infty$ . But then, by stationarity,

$$\begin{aligned} \mathbb{P}[\Delta F_s(k) = \underline{0}] &\geq \mathbb{P}[F_s(k-m-1) \leq m, \sigma_{k'} > s \ \forall k-m-1 < k' \leq k] \\ &= \mathbb{P}[F_s(k-m-1) \leq m] \cdot \mathbb{P}[\sigma_{k'} > s \ \forall k-m-1 < k' \leq k] > 0, \end{aligned} \quad (31)$$

where we have used that  $F_s(k-m-1)$  is a function of  $(\sigma_{k'})_{k' \leq k-m-1}$  and hence independent of  $(\sigma_{k'})_{k-m-1 < k' \leq k}$ . This proves (29).

The idea of the proof of (30) is easily explained. For each  $k$  such that  $s < \sigma_k \leq t$ , at most two  $\underline{0}$ 's are destroyed, so the density of  $\underline{0}$ 's can at most decrease by two times the density of such  $k$ 's, i.e., by  $2(e^{-s} - e^{-t})$ . To make this precise, let us define

$$\Delta G_t(k) := 21\{\sigma_k \in (s, t]\} - 1\{\Delta F_s(k) = \underline{0}\} \quad (k \in \mathbb{Z}), \quad (32)$$

and let  $G_t : \mathbb{Z} \rightarrow \mathbb{Z}$  be defined by

$$G_t(0) = 0 \quad \text{and} \quad G_t(k) - G_t(k-1) = \Delta G_t(k) \quad (k \in \mathbb{Z}). \quad (33)$$

We say that  $k \in \mathbb{Z}$  is a *point of decrease* of the function  $G_t$  if

$$G_t(k') > G_t(k) \quad \forall k' < k. \quad (34)$$

Let  $t \geq s$  be such that  $2(e^{-s} - e^{-t}) < \varepsilon$ . It follows from Lemma 8 below that

$$\mathbb{P}[k \text{ is a point of decrease of } G_t] = -\mathbb{E}[\Delta G_t(0)] = \varepsilon - 2(e^{-s} - e^{-t}), \quad (35)$$

as long as the expression on the right-hand side is positive. We will prove that there exists a solution  $(\tilde{Y}_k^{(t)})_{k \in \mathbb{Z}}$  of the two-way infinite inductive relation (27) taking values in the finite subsets of  $[0, t]$  such that the associated function  $\tilde{F}_t(k) := |\tilde{Y}_k^{(t)}|$  satisfies

$$\Delta \tilde{F}_t(k) = \underline{0} \text{ for each point of decrease } k \text{ of } G_t. \quad (36)$$

By Lemma 6, we have  $F_t \leq \tilde{F}_t$ , so it follows that  $F_t(k)$  is a.s. finite for each  $k \in \mathbb{Z}$ , and by (35) we see that

$$\mathbb{P}[\Delta F_t(k) = \underline{0}] \geq \mathbb{P}[\Delta \tilde{F}_t(k) = \underline{0}] \geq \mathbb{P}[k \text{ is a point of decrease of } G_t] = \varepsilon - 2(e^{-s} - e^{-t}), \quad (37)$$

proving (30).

We are left with the task of proving (36). By (35) and the ergodicity of the random variables  $(\sigma_k)_{k \in \mathbb{Z}}$ , points of decrease of  $G_t$  occur with spatial density  $\varepsilon - 2(e^{-s} - e^{-t})$ , which is positive by assumption. In particular, there exists sequences of such points tending to  $-\infty$  and  $+\infty$ .

Let  $m$  and  $n$  be points of decrease of  $G_t$  that are consecutive in the sense that  $m < n$  and  $\{m_1, \dots, n-1\}$  does not contain any points of decrease of  $G_t$ . Let  $(\tilde{Y}_k^{(t)})_{k \geq m}$  be defined by the inductive relation (27) with  $\tilde{Y}_m^{(t)} = \emptyset$ . We claim that

$$|\tilde{Y}_k^{(t)} \cap (s, t]| \leq G_t(k) - G_t(m) \quad (m \leq k < n). \quad (38)$$

Indeed, in a given step  $k$ , the left-hand side of this equation can only increase by one if  $\sigma_k \in (s, t]$ , but in this case  $G_t$  also increases by one. The right-hand side decreases by one if  $\Delta F_s(k) = 0$  and  $\sigma_k \notin (s, t]$ , but in this case either the left-hand side also decreases by one, or it is already zero. Since  $m$  and  $n$  are consecutive points of decrease of  $G_t$ , we have  $G_t(k) - G_t(m) \geq 0$  for all  $m \leq k < n$ , so also in this case the order is preserved.

Since  $n$  is a point of decrease of  $G_t$ , we have  $\Delta F_s(n) = 0$  and  $\sigma_n \notin (s, t]$ , so (38) proves that  $\tilde{F}_t(k) := |\tilde{Y}_k^{(t)}|$  satisfies  $\Delta \tilde{F}_t(n) = \underline{0}$ . Pasting together solutions of (27) between consecutive points of decrease of  $G_t$ , we obtain a two-way infinite solution satisfying (36). ■

**Lemma 8 (Points of increase)** *Let  $\omega = (\omega_k)_{k \in \mathbb{Z}}$  be an i.i.d. collection of random variables taking values in a measurable space  $(E, \mathcal{E})$ . Let  $\phi : E^{\mathbb{Z}} \rightarrow \mathbb{Z}$  be a measurable function, let  $\theta_n : E^{\mathbb{Z}} \rightarrow E^{\mathbb{Z}}$  be the shift operator  $(\theta_n u)_k := u_{k+n}$ , and let  $(G_k)_{k \in \mathbb{Z}}$  be the unique integer-valued process such that  $G_0 = 0$  and*

$$G_k - G_{k-1} = \phi(\theta_k \omega) \quad (k \in \mathbb{Z}). \quad (39)$$

*Assume that  $\mathbb{E}[|G_1 - G_0|] < \infty$ ,  $\mathbb{E}[G_1 - G_0] > 0$ , and  $G_1 - G_0 \leq 1$  a.s. Then*

$$\lim_{n \rightarrow \infty} n^{-1} \sum_{k=1}^n 1_{\{G_m > G_k \ \forall m > k\}} = \mathbb{P}[G_m > G_0 \ \forall m > 0] = \mathbb{E}[G_1 - G_0] \quad \text{a.s.} \quad (40)$$

**Proof** We call

$$A_k := \{G_m > G_k \ \forall m > k\} \quad (41)$$

the event that  $k \in \mathbb{Z}$  is a point of increase of  $G$ . The first equality in (40) follows from Birkhoff's ergodic theorem and the ergodicity of the i.i.d. random variables  $(\omega_k)_{k \in \mathbb{Z}}$ . Ergodicity also implies that

$$\lim_{n \rightarrow \infty} n^{-1} \sum_{k=1}^n G_k = \mathbb{E}[G_1 - G_0], \quad (42)$$

which is positive by assumption, so a.s.  $G$  has infinitely many points of increase. For each  $k \in \mathbb{Z}$ , let

$$\kappa(k) := \min\{k' \geq k : G_m > G_{k'} \text{ for all } m > k'\}, \quad (43)$$

be the position of the next point of increase to the right, which is a.s. well-defined. Set  $f(k, k') := \mathbb{E}[(G_k - G_{k-1})1_{\{\kappa(k)=k'\}}]$ . We claim that

$$\begin{aligned} \mathbb{E}[G_1 - G_0] &\stackrel{1}{=} \mathbb{E}[(G_1 - G_0) \left( \sum_{k' \in \mathbb{Z}} 1_{\{\kappa(1)=k'\}} \right)] \stackrel{2}{=} \sum_{k' \in \mathbb{Z}} f(1, k') \stackrel{3}{=} \sum_{k \in \mathbb{Z}} f(k, 0) \\ &\stackrel{4}{=} \mathbb{E} \left[ \sum_{k \in \mathbb{Z}} (G_k - G_{k-1}) 1_{\{\kappa(k)=0\}} \right] \stackrel{5}{=} \mathbb{P}[G_m > G_0 \text{ for all } m > 0]. \end{aligned} \quad (44)$$

Here equality 1 follows from the fact that  $\kappa(0)$  is a.s. well-defined, 2 follows from Fubini and the assumption that  $\mathbb{E}[|G_1 - G_0|] < \infty$ , and 3 follows from the fact that  $f(0, k) = f(-k, 0)$  and a change of the summation order. Equation 4 is again Fubini, using the fact that

$$\sum_{k \in \mathbb{Z}} \mathbb{E}[(G_k - G_{k-1})1_{\{\kappa(k)=0\}}] = \sum_{k' \in \mathbb{Z}} \mathbb{E}[(G_1 - G_0)1_{\{\kappa(1)=k'\}}] < 0. \quad (45)$$

To see why equation 5 holds, we observe that if  $m$  is a point of increase of  $G$ , then  $G_{m+1} = G_m + 1$  by our assumption that increments of  $G$  are a.s.  $\leq 1$ , and hence  $m' := \max\{k : G_k = G_m + 1\}$  is the next point of increase after  $m$ . Letting  $l := \max\{k < 0 : k \text{ is a point of increase of } G\}$ , we have

$$\begin{aligned} &\sum_{k \in \mathbb{Z}} (G_k - G_{k-1}) 1_{\{\kappa(k)=0\}} \\ &= 1_{\{G_m > G_0 \text{ for all } m > 0\}} \sum_{k=l+1}^0 (G_k - G_{k-1}) = 1_{\{G_m > G_0 \text{ for all } m > 0\}}, \end{aligned} \quad (46)$$

where we have used that  $G_0 - G_1 = 1$  on the event that 0 is a point of increase of  $G$ . ■

Our next aim is to derive the differential equation (13). We will actually derive the somewhat different equation (47) below, but it is easy to check that the two equations have the same solution (14) with the initial conditions  $\mathbb{P}[\Delta F_t(0) = 0] = 1$ ,  $\mathbb{P}[\Delta F_t(0) = -1] = 0$ .

**Proposition 9 (Differential equation)** *Let  $F_t$  be the as in (23) and assume that  $u > 0$  is such that  $F_u(k) < \infty$  a.s. for all  $k \in \mathbb{Z}$ . For  $t \in [0, u]$  define  $\Delta F_t(k)$  as in (11). Then the functions  $t \mapsto \mathbb{P}[\Delta F_t(0) = 0]$  and  $t \mapsto \mathbb{P}[\Delta F_t(0) = -1]$  are continuously differentiable on  $[0, u]$  and satisfy the differential equations*

$$\begin{aligned} \frac{\partial}{\partial t} \mathbb{P}[\Delta F_t(k) = 0] &= -\mathbb{P}[\Delta F_t(k) = 0] - e^{-t}, \\ \frac{\partial}{\partial t} \mathbb{P}[\Delta F_t(k) = -1] &= -\mathbb{P}[\Delta F_t(k) = -1] + e^{-t}. \end{aligned} \quad (47)$$

**Proof** For all  $t \in [0, u]$  and  $k \in \mathbb{Z}$ , define  $\kappa_t(k)$  as in (8), i.e.,

$$\kappa_t(k) := \inf\{k' > k : \Delta F_{t-}(k') = \underline{0}\}. \quad (48)$$

Since the set  $\{k \in \mathbb{Z} : \Delta F_t(k) = \underline{0}\}$  is nonincreasing in  $t$ , we see that for each  $k \in \mathbb{Z}$  there is at most one  $t \in [0, u]$  such that

$$\text{at time } t, \text{ one has } t = \sigma_{k'} \text{ for some } k' < k \text{ and } \kappa_t(k') = k. \quad (49)$$

Note that at such a time,  $\Delta F_{t-}(k) = \underline{0}$  and  $\Delta F_t = -1$ . If such a time exists, we denote it by  $\tau_k$ . For definiteness, we set  $\tau_k := \infty$  if there exists no  $t \in [0, u]$  such that (49) holds.

For all  $0 \leq s < t \leq u$ , we observe that

$$\begin{aligned} \mathbb{P}[\Delta F_t(k) = \underline{0}] &= \mathbb{P}[\Delta F_s(k) = \underline{0}, \{\sigma_k, \tau_k\} \cap (s, t] = \emptyset], \\ \mathbb{P}[\Delta F_t(k) = -1] &= \mathbb{P}[\Delta F_s(k) = -1, \sigma_k \notin (s, t]] + \mathbb{P}[\tau_k \in (s, t], \sigma_k > t]. \end{aligned} \quad (50)$$

Since  $\Delta F_s(k) = -1$  implies that  $\sigma_k > s$  and  $\Delta F_s$  depends only on information about the times  $(\sigma_{k'})_{k' \in \mathbb{Z}}$  until time  $s$ , we have

$$\mathbb{P}[\Delta F_s(k) = -1, \sigma_k \notin (s, t]] = \mathbb{P}[\Delta F_s(k) = -1]e^{-(t-s)}. \quad (51)$$

Using translation invariance and changing the summation order, we see that

$$\begin{aligned} \mathbb{P}[\tau_k \in (s, t]] &= \sum_{k' \in \mathbb{Z}} \mathbb{P}[\sigma_{k'} \in (s, t], \kappa_{\sigma_{k'}}(k') = k] \\ &= \sum_{k' \in \mathbb{Z}} \mathbb{P}[\sigma_0 \in (s, t], \kappa_{\sigma_0}(0) = k - k'] = \mathbb{P}[\sigma_0 \in (s, t]] = e^{-s} - e^{-t} = e^{-s}(1 - e^{-(t-s)}). \end{aligned} \quad (52)$$

Finally, we observe that  $\tau_k \in (s, t]$  implies  $\tau_k < \sigma_k$  and we use this to estimate

$$\mathbb{P}[\tau_k \in (s, t] \text{ and } \sigma_k \in (s, t]] \leq \mathbb{P}[\tau_k \in (s, t] \text{ and } \sigma_k - \tau_k \leq (t - s)] = e^{-s}(1 - e^{-(t-s)})^2, \quad (53)$$

where in the last step we have used (52) and the fact that by the memoryless property of the exponential distribution, conditional on  $\tau_k \in (s, t]$ , the random variable  $\sigma_k - \tau_k$  is exponentially distributed.

We rewrite the right-hand side of the first equation in (50) as

$$\begin{aligned} &\mathbb{P}[\Delta F_s(k) = \underline{0}, \{\sigma_k, \tau_k\} \cap (s, t] = \emptyset] \\ &= \mathbb{P}[\Delta F_s(k) = \underline{0}] - \mathbb{P}[\Delta F_s(k) = \underline{0}, \sigma_k \in (s, t]] - \mathbb{P}[\Delta F_s(k) = \underline{0}, \tau_k \in (s, t]] \\ &\quad + \mathbb{P}[\Delta F_s(k) = \underline{0}, \sigma_k \in (s, t] \text{ and } \tau_k \in (s, t]]. \end{aligned} \quad (54)$$

Here, arguing as in (51) and using (52)

$$\begin{aligned} \mathbb{P}[\Delta F_s(k) = \underline{0}, \sigma_k \in (s, t]] &= \mathbb{P}[\Delta F_s(k) = \underline{0}](1 - e^{-(t-s)}), \\ \mathbb{P}[\Delta F_s(k) = \underline{0}, \tau_k \in (s, t]] &= \mathbb{P}[\tau_k \in (s, t]] = 1 - e^{-(t-s)}. \end{aligned} \quad (55)$$

Using also (53), it follows that for any  $0 \leq s < s + \varepsilon \leq u$ ,

$$\mathbb{P}[\Delta F_{s+\varepsilon}(k) = \underline{0}] - \mathbb{P}[\Delta F_s(k) = \underline{0}] = -\varepsilon \mathbb{P}[\Delta F_s(k) = \underline{0}] - \varepsilon e^{-s} + O(\varepsilon^2), \quad (56)$$

where  $O(\varepsilon^2)$  is a term that can uniformly be estimated as  $|O(\varepsilon^2)| \leq K\varepsilon^2$  for some fixed  $K < \infty$ . Treating the second equation in (50) in a similar manner, using (51), (52) and (53), we obtain

$$\mathbb{P}[\Delta F_{s+\varepsilon}(k) = -1] - \mathbb{P}[\Delta F_s(k) = -1] = -\varepsilon \mathbb{P}[\Delta F_s(k) = -1] + \varepsilon e^{-s} + O(\varepsilon^2). \quad (57)$$

Letting  $\varepsilon \downarrow 0$ , we arrive at (47).  $\blacksquare$

**Proof of Theorem 4** Let  $I$  be the set of all  $t \in [0, \infty)$  such that case (i) of Lemma 5) holds, i.e.,  $F_t(k) < \infty$  a.s. for all  $k \in \mathbb{Z}$ . As remarked above Proposition 7, we have  $0 \in I$ . It is immediate from the definition of  $F_t(k)$  in (23) that  $t \mapsto F_t(k)$  is a.s. nondecreasing, so  $I$  is a subinterval of  $[0, \infty)$  containing 0. It follows from Proposition 7 that  $I$  is of the form  $I = [0, t_c)$  for some  $t_c \in (0, \infty]$ . Moreover,  $\mathbb{P}[\Delta F_t(k) = \underline{0}] > 0$  for  $t \in [0, t_c)$  and if  $t_c < \infty$ , then we must have

$$\lim_{t \uparrow t_c} \mathbb{P}[\Delta F_t(k) = \underline{0}] = 0. \quad (58)$$

Solving the differential equation of Proposition 9, we see that

$$\begin{aligned} \mathbb{P}[\Delta F_t(k) = \underline{0}] &= (1-t)e^{-t}, \\ \mathbb{P}[\Delta F_t(k) = -1] &= te^{-t}, \end{aligned} \quad (t \in [0, t_c), k \in \mathbb{Z}), \quad (59)$$

which together with (58) allows us to conclude that  $t_c = 1$ .

Since  $F_1(k) = \infty$  a.s., it follows that  $N_k := \inf(Y_k) < 1$  a.s., and since  $F_t(k) < \infty$  a.s. for all  $t < 1$ , this infimum is in fact a minimum. Since for each  $t < 1$ , the restricted process  $Y^{(t)}$  solves the inductive relation (27) and since for each  $k$ , there exists a  $t < 1$  such that  $N_k^{(t)} < t$ , it is easy to see that  $(Y_k)_{k \in \mathbb{Z}}$  solves the inductive relation (6).

Stationarity implies (see Lemma 10 below) that

$$\mathbb{P}[\Delta F_t(k) = +1] = \mathbb{P}[\Delta F_t(k) = -1] \quad (0 \leq t < 1), \quad (60)$$

which together with (59) and the requirement that the total probability is one yields (25).  $\blacksquare$

**Lemma 10 (Stationary increments)** *Let  $(F(k))_{k \in \mathbb{Z}}$  be a stationary process, and assume that  $\mathbb{E}[|F(1) - F(0)|] < \infty$ . Then  $\mathbb{E}[F(1) - F(0)] = 0$ .*

**Proof** For  $M > 0$ , let  $F^M(k) := F(k)$  if  $-M \leq F(k) \leq M$  and  $F^M(k) := M$  or  $-M$  if  $F(k) \geq M$  or  $F(k) \leq -M$ , respectively. By stationarity,  $\mathbb{E}[F^M(1)] = \mathbb{E}[F^M(0)]$  and hence  $\mathbb{E}[F^M(1) - F^M(0)] = 0$ . Letting  $M \uparrow \infty$ , using the fact that  $|F^M(1) - F^M(0)| \leq |F(1) - F(0)|$  and dominated convergence, we conclude that  $\mathbb{E}[F(1) - F(0)] = 0$ .  $\blacksquare$

**Remark** Although, by Theorem 4, the function  $F_t$  is finite only for  $t < 1$ , it is possible to give a sensible definition of  $\Delta F_t$  also for  $t \geq 1$  by setting (compare (11) and Figure 1)

$$\Delta F_t(k) := \begin{cases} \underline{0} & \text{if } t < N_{k-1} < \sigma_k, \\ \bar{0} & \text{if } N_{k-1} < \sigma_k \leq t \\ -1 & \text{if } N_{k-1} \leq t < \sigma_k, \\ +1 & \text{if } \sigma_k \leq t \wedge N_{k-1}. \end{cases} \quad (k \in \mathbb{Z}, t \in [0, 1]), \quad (61)$$

where  $N_k := \min(Y_k)$  ( $k \in \mathbb{Z}$ ) and  $(Y_k)_{k \in \mathbb{Z}}$  is the lower invariant process.

## 2.4 Ergodicity

In the present section, we use Theorem 4 to derive Theorems 1 and 2. Most of the work is already done. The remaining arguments are for a large part standard Markov chain theory.

Let  $Y_0 = y$  be any finite subset of  $[0, \infty)$  and let  $(Y_k)_{k \geq 0}$  be the Markov chain with initial state  $Y_0$  defined by the inductive relation (6). We have already seen that for each  $t \in [0, \infty)$ , the restricted process  $Y_k^{(t)} := Y_k \cap [0, t]$  is a Markov chain and in fact given by the inductive relation (27). When we need to specify the initial state  $y$ , we make this explicit in our notation by writing  $(Y_k^y)_{k \geq 0}$  for the original process and  $(Y_k^{y(t)})_{k \geq 0}$  for the restricted process. As in (3), we define

$$\tilde{\tau}_t^y := \inf\{k > 0 : Y_k^{y(t)} = \emptyset\}. \quad (62)$$

In particular, if  $y = \emptyset$ , then this is the first return time of the restricted process to the empty configuration.

**Lemma 11 (Expected return time)** *One has  $\mathbb{E}[\tilde{\tau}_t^\emptyset] = (1 - t)^{-1}$  for each  $t \in [0, 1)$ .*

**Proof** Let  $(Y_k)_{k \in \mathbb{Z}}$  be the lower invariant process from (22) and let  $F_t$  and  $\Delta F_t$  be defined as in (23) and (11). Then, by Theorem 4, for  $t \in [0, 1)$  one has

$$(1 - t)e^{-t} = \mathbb{P}[\Delta F_t(k) = \underline{0}] = \mathbb{P}[F_t(k - 1) = 0] \mathbb{P}[\sigma_k > t] = \mathbb{P}[F_t(k - 1) = 0]e^{-t}, \quad (63)$$

which proves that the restricted lower invariant process satisfies  $\mathbb{P}[Y_k^{(t)} = \emptyset] = 1 - t$ . Let  $\lambda_t(k) := \sup\{k' < k : Y_{k'}^{(t)} = \emptyset\}$  ( $k \in \mathbb{Z}$ ). Then, for  $t \in [0, 1)$ , by the mass transport principle,

$$1 = \sum_{k \in \mathbb{Z}} \mathbb{P}[\lambda_t(0) = -k] = \sum_{k \in \mathbb{Z}} \mathbb{P}[\lambda_t(k) = 0] = \sum_{k \in \mathbb{Z}} \mathbb{P}[Y_0^{(t)} = \emptyset, 0 < \tilde{\tau}_t \leq k] = (1 - t)\mathbb{E}[\tilde{\tau}_t^\emptyset]. \quad (64)$$

■

**Lemma 12 (Transience)** *One has  $\mathbb{P}[\tilde{\tau}_t^\emptyset = \infty] > 0$  for each  $t \in (1, \infty)$ .*

**Proof** Let  $F_t(k) := |Y_k^\emptyset \cap [0, t]|$  ( $k \geq 0$ ) and for  $k \geq 1$  define  $\Delta F_t(k)$  as in (11). We observe that for any  $0 \leq s < t$ ,

$$F_t(n) \geq \sum_{k=1}^n 1_{\{s < \sigma_k < t\}} - \sum_{k=1}^n 1_{\{F_s(k-1) = 0\}}, \quad (65)$$

where we have used that  $\Delta F_t(k) \geq 0$  whenever  $s < \sigma_k < t$ , and  $\Delta F_t(k) = +1$  if moreover  $F_s(k-1) \neq 0$ . Since the process  $(F_s(k))_{k \geq 0}$  makes i.i.d. excursions from 0 with length distributed as  $\tilde{\tau}_s^\emptyset$ , by Lemma 11 and the strong law of large numbers, we have for each  $s \in [0, 1)$

$$n^{-1} \sum_{k=1}^n 1_{\{s < \sigma_k < t\}} \xrightarrow[n \rightarrow \infty]{} (e^{-s} - e^{-t}) \quad \text{and} \quad n^{-1} \sum_{k=1}^n 1_{\{F_s(k-1) = 0\}} \xrightarrow[n \rightarrow \infty]{} 1 - s \quad \text{a.s.} \quad (66)$$

Choosing  $s$  close enough to 1 such that  $1 - s < e^{-s} - e^{-t}$ , we see that  $F_t(n) \rightarrow \infty$  a.s. Since  $(F_t(k))_{k \geq 0}$  makes i.i.d. excursions from 0 with length distributed as  $\tilde{\tau}_t^\emptyset$ , it follows that  $\mathbb{P}[\tilde{\tau}_t^\emptyset = \infty] > 0$ . ■

Lemmas 11 and 12 show that the restricted process  $Y_k^{(t)} = Y_k \cap [0, t]$  started from the empty configuration returns to the empty configuration in finite expected time if  $t < 1$  and

has a positive probability never to return to the empty configuration if  $t > 1$ . We would like to conclude from this that the process, started in an arbitrary initial state, is “positively recurrent” for  $t < 1$  and “transient” for  $t > 1$ . Since the state space of  $Y^{(t)}$  is uncountable, we have to specify in exactly which meaning we use these terms. Recall from (62) that  $\tilde{\tau}_t^y$  is the first time after time zero that the restricted process  $Y^{y(t)}$  started in the initial state  $y$  is in the empty state. We will say that  $Y^{(t)}$  is *positive recurrent*, *null recurrent*, or *transient* depending on whether case (i), (ii), or (iii) of the following lemma occurs.

**Lemma 13 (Recurrence versus transience)** *For each  $t > 0$ , exactly one of the following three possibilities occurs.*

- (i) *For all finite  $y \subset [0, \infty)$ , one has  $\mathbb{E}[\tilde{\tau}_t^y] < \infty$ .*
- (ii) *For all finite  $y \subset [0, \infty)$ , one has  $\tilde{\tau}_t^y < \infty$  a.s. and  $\mathbb{E}[\tilde{\tau}_t^y] = \infty$ .*
- (iii) *For all finite  $y \subset [0, \infty)$ , one has  $\mathbb{P}[\tilde{\tau}_t^y = \infty] > 0$ .*

We first need a preparatory result. Instead of using general Markov chain techniques (such as the theory of Harris recurrence) to prove Lemma 13, we will rely on monotonicity arguments that are special to our model. The next lemma, which is of some interest in its own right, may loosely be described as saying that for  $Y^{(t)}$  to avoid becoming the empty set, it is good to have many particles that are situated as far as possible to right in the interval  $[0, t]$ .

**Lemma 14 (Second comparison lemma)** *For each  $0 \leq s \leq t$  and finite  $y \subset [0, \infty)$ , let  $F_{s,t}^y(k) := |Y_k^y \cap [s, t]|$  ( $k \geq 0$ ). Fix  $t > 0$  and let  $x, y \subset [0, \infty)$  be finite. Then*

$$F_{s,t}^x(0) \leq F_{s,t}^y(0) \quad \forall s \in [0, t] \quad \text{implies} \quad F_{s,t}^x(k) \leq F_{s,t}^y(k) \quad \forall s \in [0, t], \quad k \geq 0. \quad (67)$$

**Proof** It suffices to prove (67) for  $k = 1$ ; the general statement follows by induction. Without loss of generality, we may also assume that  $x$  and  $y$  are subsets of  $[0, t]$ . Order the elements of  $x$  and  $y$  as  $x = \{x_1, \dots, x_n\}$  and  $y = \{y_1, \dots, y_m\}$  with  $x_n < \dots < x_1$  (in this order!) and  $y_m < \dots < y_1$ . Then the assumption that  $F_{s,t}^x(0) \leq F_{s,t}^y(0) \quad \forall s \in [0, t]$  is equivalent to the statement that  $m \geq n$  and  $x_i \leq y_i$  for all  $i = 1, \dots, n$ . We must show that we can order the elements of  $\tilde{x} := Y_1^x \cap [0, t]$  and  $\tilde{y} := Y_1^y \cap [0, t]$  in the same way. We distinguish three different cases.

Case I:  $\sigma_1 < x_n$ . In this case, no points are removed from  $x$  while  $\tilde{x}_{n+1} := \sigma_1$  is added as the  $(n+1)$ -th element. Since  $x_n \leq y_n$ , the elements  $y_1, \dots, y_n$  remain unchanged while  $\tilde{y}_{n+1}$  is the minimal element of  $\{\sigma_1\} \cup \{y_m, \dots, y_{n+1}\}$ , which lies on the right of  $\tilde{x}_{n+1} = \sigma_1$ .

Case II:  $x_n < \sigma_1 < t$ . In this case,  $x_n$  is removed from  $x$  and there exist  $1 \leq n' \leq n$  and  $n' \leq m' \leq m+1$  such that  $\sigma_1$  is inserted into  $x$  between the  $n'$ -th and  $(n'-1)$ -th element and into  $y$  between the  $m'$ -th and  $(m'-1)$ -th element, where we allow for the cases that  $n' = 1$  ( $\sigma_1$  is added at the right end of  $x$  and possibly also of  $y$ ) and  $m' = m+1$  ( $\sigma_1$  is added at the left end of  $y$ ). The elements of the new sets  $\tilde{x}$  and  $\tilde{y}$ , ordered from low to high, are now

$$\begin{aligned} \{x_{n-1}, \dots, x_{m'}, x_{m'-1}, \dots, x_{n'}, \sigma_1, x_{n'-1}, \dots, x_1\} &= \tilde{x}, \\ \{y_{m-1}, \dots, y_n, y_{n-1}, \dots, y_{m'}, \sigma_1, y_{m'-1}, \dots, y_{n'}, y_{n'-1}, \dots, y_1\} &= \tilde{y}. \end{aligned} \quad (68)$$

Here  $x_{n-1}, \dots, x_{m'}$  lie on the left of  $y_{n-1}, \dots, y_{m'}$ , and likewise  $x_{n'-1}, \dots, x_1$  lie on the left of  $y_{n'-1}, \dots, y_1$ , respectively. Since moreover

$$x_{m'-1} < \dots < x_{n'} < \sigma_1 < y_{m'-1} < \dots < y_{n'}, \quad (69)$$

these elements are ordered in the right way too.

Case III:  $t < \sigma_1$ . In this case, the lowest elements of  $x$  and  $y$  are removed while no new elements are added, which obviously also preserves the order. ■

**Proof of Lemma 13** It suffices to show that for any finite  $y \in [0, \infty)$ , one has  $\tilde{\tau}_t^y < \infty$  a.s. if and only if  $\tilde{\tau}_t^\emptyset < \infty$  a.s., and likewise,  $\mathbb{E}[\tilde{\tau}_t^y] = \infty$  if and only if  $\mathbb{E}[\tilde{\tau}_t^\emptyset] = \infty$ . By the first comparison Lemma 3,  $\tilde{\tau}_t^\emptyset \leq \tilde{\tau}_t^y$  which immediately gives us the implications in one direction. To complete the proof, we must show that  $\mathbb{P}[\tilde{\tau}_t^y = \infty] > 0$  implies  $\mathbb{P}[\tilde{\tau}_t^\emptyset = \infty] > 0$  and  $\mathbb{E}[\tilde{\tau}_t^y] = \infty$  implies  $\mathbb{E}[\tilde{\tau}_t^\emptyset] = \infty$ .

Recalling notation introduced in the second comparison Lemma 14, we observe that for any finite  $y \in [0, \infty)$  there is a  $k \geq 1$  such that

$$\mathbb{P}[\tilde{\tau}_t^\emptyset > k \text{ and } F_{s,t}^\emptyset(k) \geq F_{s,t}^y(0) \forall s \in [0, t]] > 0. \quad (70)$$

Using this, Lemma 14, and the Markov property, we see that

$$\begin{aligned} \mathbb{P}[\tilde{\tau}_t^\emptyset = \infty] &\geq \mathbb{P}[\tilde{\tau}_t^\emptyset > k \text{ and } F_{s,t}^\emptyset(k) \geq F_{s,t}^y(0) \forall s \in [0, t]] \mathbb{P}[\tilde{\tau}_t^y = \infty], \\ \mathbb{E}[\tilde{\tau}_t^\emptyset] &\geq \mathbb{P}[\tilde{\tau}_t^\emptyset > k \text{ and } F_{s,t}^\emptyset(k) \geq F_{s,t}^y(0) \forall s \in [0, t]] (k + \mathbb{E}[\tilde{\tau}_t^y]), \end{aligned} \quad (71)$$

which together with (70) gives us the desired implications. ■

Positive recurrence in the sense of Lemma 13 suffices to prove ergodicity of the restricted process  $Y^{(t)}$ . In fact, the following general result applies.

**Proposition 15 (Markov chain with an atom)** *Let  $P$  be a measurable probability kernel on a Polish space  $E$  and for each  $x \in E$ , let  $(X_k^x)_{k \geq 0}$  denote the Markov chain with initial state  $x$  and transition kernel  $P$ . Let  $z \in E$  be fixed and let*

$$\tau^x := \inf\{k > 0 : X_k^x = z\} \quad (x \in E). \quad (72)$$

*Assume that  $\mathbb{E}[\tau^z] < \infty$ ,  $\mathbb{P}[\tau^x < \infty] = 1$  for all  $x \in E$ , and that the greatest common divisor of  $\{k > 0 : \mathbb{P}[\tau^z = k] > 0\}$  is one. Then there exists a unique invariant law  $\nu$  for  $P$  and*

$$\|\nu - \mathbb{P}[X_k^x \in \cdot]\| \xrightarrow[k \rightarrow \infty]{} 0, \quad (73)$$

*where  $\|\cdot\|$  denotes the total variation norm.*

**Proof** This follows from standard arguments, so we only sketch the proof. First, one can check that

$$\nu := \mathbb{E}[\tau^z]^{-1} \sum_{k=1}^{\infty} \mathbb{P}[\tau^z \leq k \text{ and } X^z \in \cdot] \quad (74)$$

is an invariant law for  $P$ . We can couple the corresponding stationary process  $(X_k)_{k \geq 0}$  and the process  $(X_k^x)_{k \geq 0}$  started in a deterministic initial state  $x$  in such a way that they evolve independently until the time  $\sigma := \inf\{k \geq 0 : X_k^x = z = X_k\}$ . Since  $\mathbb{P}[\tau^x < \infty] = 1$  for all  $x \in E$ , both processes reach  $z$  in a finite random time and after that make i.i.d. excursions away from  $z$  whose length has finite mean  $\mathbb{E}[\tau^z]$ . Using also the aperiodicity assumption, it follows that  $\sigma < \infty$  a.s. so the coupling is successful. ■

**Remark 1** The assumption that the state space is Polish guarantees that Kolmogorov's extension theorem can be applied to construct the process from its finite dimensional distributions.



This assumption can certainly be relaxed; see [MT09, Section 3.1] for a “general” set-up which is, however, so general that singletons  $\{z\}$  may fail to be measurable. When we apply Proposition 15 below to the restricted process  $Y^{(t)}$ , the state space is the set of all simple counting measures on  $[0, t]$ , equipped with the topology of weak convergence. This space is Polish because of the following facts: 1. for any Polish space  $E$ , the space  $\mathcal{M}(E)$  of finite measures on  $E$ , equipped with the topology of weak convergence, is Polish, 2. the set  $\mathcal{N}[0, t]$  of all counting measures on  $[0, t]$  is a closed subset of  $\mathcal{M}[0, t]$ , 3. the set of all simple counting measures is a  $G_\delta$ -subset of  $\mathcal{N}[0, t]$ , 4. a  $G_\delta$ -subset of a Polish space is Polish [Bou58, §6 No. 1, Theorem. 1].

**Remark 2** The fact that formula (74) defines an invariant law follows from [MT09, Theorem 10.1.2 (i)]. The fact that our coupling is successful follows from [Woe09, Lemma 3.46]. The latter is written down for Markov chains with countable state space only, but this applies generally since any  $\mathbb{N}_+$ -valued random variable with finite mean is the law of the return time of a suitably constructed positively recurrent Markov chain with countable state space.

**Proof of Theorem 1** By Lemma 11, for each  $t < 1$ , the restricted process  $Y_k^{y^{(t)}} = Y_k^y \cap [0, t]$  is positively recurrent in the sense of Lemma 13, case (i), so a.s.  $Y_k^{y^{(t)}} = \emptyset$  for infinitely many  $k$ , proving that

$$\limsup_{k \rightarrow \infty} N_k^{y^{(t)}} \geq t \quad \text{a.s.} \quad (t < 1), \quad (75)$$

where  $N_k^{(t)} := \inf(Y_k^{y^{(t)}} \cup \{t\})$  ( $k \in \mathbb{Z}$ ). On the other hand, by Lemma 12,  $Y^y$  is transient in the sense of Lemma 13, case (i), for all  $t > 1$ , so

$$\mathbb{P}[\exists n \text{ s.t. } Y_k^y \cap [0, t] \neq \emptyset \quad \forall k \geq n] = 1. \quad (76)$$

Combining this with (75) we see that

$$\limsup_{k \rightarrow \infty} N_k^y = 1 \quad \text{a.s.}, \quad (77)$$

where  $N_k^y := \min(Y_k^y \cup \{\infty\})$  ( $k \geq 0$ ). Translating this to the process  $X$  through the transformaton  $q = 1 - e^{-t}$  as discussed in Section 2.1 yields Theorem 1. ■

**Proof of Theorem 2** Formula (4) is just the translation of Lemmas 11 and 12 to the process  $X$  through the transformaton  $t = -\log(1 - q)$  as discussed in Section 2.1.

Let  $(Y_k)_{k \in \mathbb{Z}}$  denote the lower invariant process from (22). By Theorem 4, for each  $t < 1$ , setting  $\nu := \mathbb{P}[Y_0 \cap [0, t] \in \cdot]$  defines an invariant law for the process restricted to  $[0, t]$ . By Lemma 11, this process is positively recurrent in the sense of Lemma 13, case (i). Since  $\mathbb{P}[\tilde{\tau}_t^\emptyset = 1] = \mathbb{P}[\sigma_1 > t] = e^{-t}$ , this process is ergodic in the sense of Proposition 15, i.e.,  $\nu$  is its unique invariant law and the long-time limit law (w.r.t. the total variation norm) started from any initial state. Translated for the process  $X$ , this yields (5). It has been proved in Theorem 4 that  $Y_0 \cap [0, 1)$  is a.s. an infinite set, so the same is true for the set  $X_\infty$  which is the image of  $Y_0 \cap [0, 1)$  under the map  $t \mapsto 1 - e^{-t}$ . ■

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